Editorials

Chlodowsky Type (λ, q)-Bernstein Stancu Operator of Korovkin-Type Approximation Theorem of Rough I-Core of Triple Sequences

[1]Ayhan Esi, [2]Subramanian Nagarajan and [3]Kemal Ozdemir

[1]*Department of Mathematics, Malatya Turgut Ozal University, Turkey*
[2]*Department of Mathematics, Sastra University, India*
[3]*Department of Mathematics, Inonu University, Turkey*

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Corresponding Author:
Ayhan Esi
Department of Mathematics,
Malatya Turgut Ozal
University, Turkey
Email: aesi23@hotmail.com

Abstract: In this study, we obtain a Korovkin-type approximation theorem for Chlodowsky type (λ, q)-Bernstein Stancu operator of rough I-convergent of triple sequences of positive linear operators of two variables from $H_w(K)$ to $C_w(K)$. We introduce and study some basic properties of Korovkin-type approximation theorem for Chlodowsky type (λ, q)-Bernstein Stancu operator of rough I-convergent of triple sequence spaces and also study the set of all Korovkin-type approximation theorem for Chlodowsky type (λ, q)–Bernstein Stancu operator of rough I-limits of triple sequence spaces and the relation between analyticness and Korovkin-type approximation theorem for Chlodowsky type (λ, q)-Bernstein Stancu operator of rough I-core of triple sequence spaces.

Keywords: Chlodowsky Type (λ, q)-Bernstein Stancu Operator, Ideal, Triple Sequences, Rough Convergence, Closed and Convex, Cluster Points and Rough Limit Points, Korovkin-type Approximation.

Introduction

The idea of rough convergence was first introduced by (Phu, 2001; 2002; Xuan Phu, 2003) infinite-dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed, and convex; and he introduced the notion of a rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

Aytar (2008a) studied rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar (2008b) studied that the r-limit set of the sequence is equal to the intersection of these sets and that the r-core of the sequence is equal to the union of these sets. Dündar and Çakan (2014) investigated rough ideal convergence and defined the set of rough ideal limit points of a sequence the notion of I-convergence of triple sequence spaces is based on the structure of the ideal I of subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of all-natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

Our primary interest in the present paper is to obtain a general Korovkin-type approximation theorem for triple sequences of positive linear operators of two variables from $H_w(K)$ to $C_w(K)$ via statistical A-summability.

Let A be a three-dimensional summability matrix. For a given triple sequence $x = (x_{mnk})$, the A-transform of x, denoted by $Ax : x\left((Ax)_{ij\ell}\right)$, given by:

$$(Ax)_{i,j,\ell} = \sum_{(m,n,k)\in\mathbb{N}^3} a_{i,j,\ell,m,n,k}\, x_{mnk} \tag{1.1}$$

provided the triple series converges in Pringsheim's sense for every $(i,j,\ell) \in \mathbb{N}^3$.

A three dimensional matrix $A = (a_{i,j,\ell,m,n,k})$ is said to be RH-regular it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. A three dimensional matrix $A = (a_{i,j,\ell,m,n,k})$ is RH-regular if and only if:

(i) $P - \lim_{i,j} a_{i,j,\ell,m,n,k} = 0$ for each $(m, n, k) \in \mathbb{N}^3$
(ii) $P - \lim_{i,j,\ell} \sum_{(m,n,k)\in\mathbb{N}^3} a_{i,j,\ell,m,n,k} = 1$
(iii) $P - \lim_{i,j,\ell} \sum_{m\in\mathbb{N}} a_{i,j,\ell,m,n,k} = 0$ for each $n, k \in \mathbb{N}$

(iv) $P-\lim_{i,j,\ell}\sum_{n\in\mathbb{N}}a_{i,j,\ell,m,n,k}=0$ for each m, $k\in\mathbb{N}$

(v) $P-\lim_{i,j,\ell}\sum_{k\in\mathbb{N}}a_{i,j,\ell,m,n,k}=0$ for each m, $n\in\mathbb{N}$

(vi) $\sum_{(m,n,k)\in\mathbb{N}^3}\left|a_{i,j,\ell,m,n,k}\right|$ is P-convergent for every $(i,j,\ell)\in\mathbb{N}^3$

(vii) There exist finite positive integers A and B such that $\Sigma_{m,n,k>B}\left|a_{i,j,\ell,m,n,k}\right|<$

A holds for every $(i,j,\ell)\in\mathbb{N}^3$.

Now let $A=(a_{i,j,\ell,m,n,k})$ be a non-negative RH-regular summability matrix, and $K\subset\mathbb{N}^3$. Then the A-density of K is given by:

$$\delta_2^A\{K\}:=P-\lim_{i,j,\ell}\sum_{(m,n,k)\in K(\varepsilon)}a_{i,j,\ell,m,n,k}$$

where:

$$K(\epsilon):=\left\{(m,n,k)\in\mathbb{N}^3:\left|x_{mnk}-L\right|\geq\epsilon\right\}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. A real triple sequence $x=(x_{mnk})$ is said to be A-statistically convergent to a number L if, for every $\epsilon>0$:

$$\delta_2^A\left\{(m,n,k)\in\mathbb{N}^3:\left|x_{mnk}-L\right|\geq\epsilon\right\}=0.$$

In this case, we write $st_2^A-\lim_{m,n,k}=L$.

In this study, we investigate some basic properties of the Korovkin-type approximation theorem for rough I-convergence of triple sequence spaces in three-dimensional matrix spaces which are not earlier. We study the set of all rough I-limits of triple sequence spaces and also the relation between analytic ness and rough I-core of a Korovkin-type approximation theorem for triple sequence spaces. We recommend the reader to refer to (Arqub, 2015; Abu-Arqub *et al*., 2013; Al-Smadi *et al*., 2012; 2015; 2016; Momani *et al*., 2016) and (Shawagfeh *et al*., 2014) references to see the different approaches.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and let us denote the set $K_{ik\ell}=\{(m,n,k)\in K: m\geq i, n\geq j, k\geq\ell\}$. Then the natural density of K is given by:

$$\delta(K)=\lim_{i,j,\ell\to\infty}\frac{\left|K_{ij\ell}\right|}{ij\ell},$$

where, $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

In this study, we construct Chlodowsky type (λ, q)-Bernstein Stancu operators of triple sequence space is defined as:

$$B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\hat{b}_{rst,mnk}(x;q)\,f\left(\frac{[mnk]_q+\alpha}{[rst]_q+\beta}b_{rst}\right),\ (1.2)$$

where, $r,s,t\in\mathbb{N}$, $0<q\leq 1, 0\leq x\leq b_{rst}$ and b_{rst} is a sequence of positive numbers such that $\lim_{rst\to\infty}b_{rst}=\infty, \lim_{rst\to\infty}\frac{b_{rst}}{[rst]_q}=0$, :

$$\hat{b}_{rst,mnk}(x;q)=\binom{r}{m}\binom{s}{n}\binom{t}{k}\left(\frac{x}{b_{rst}}\right)^{m+n+k}\left(1-\frac{x}{b_{rst}}\right)^{(r-m)+(s-n)+(t-k)}$$

and α, $\beta\in\mathbb{R}$ and $0\leq\alpha\leq\beta$. For $\alpha=\beta=0$ we obtain the Chlodowsky type (λ, q)-Bernstein Stancu polynomials. Throughout the paper, $\mathbb{R}$ denotes the real of three-dimensional space with metric (X, d). Consider a triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operators $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ such that $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)\in\mathbb{R}, m,n,k\in\mathbb{N}$.

Let f be a continuous function defined on the closed interval [0,1]. A triple sequence of Bernstein polynomials $\left(B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\right)$ is said to be statistically convergent to 0 $\in$ R, written as $st-\lim x=0$, provided that the set:

$$K\epsilon:=\left\{(m,n,k)\in\mathbb{N}^3:\left|B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)-f(x)\right|\geq\epsilon\right\}$$

has natural density zero for any $\epsilon>0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\epsilon)=0$. That is:

$$\lim_{r,s,t\to\infty}\frac{1}{rst}\left|\left\{m\leq r,\ n\leq s,\ k\leq t:\left|B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)-(f,x)\right|\geq\epsilon\right\}\right|=0.$$

In this case, we write $\delta-\lim B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)=f(x) or B_{(r,s,t),\lambda,q}^{\alpha,\beta}(f;x)\to S_B\ f(x)$.

Throughout the paper, N denotes the set of all positive integers, χ_A-the characteristic function of $A\subset$ N, and R the set of all real numbers. A subset A of N is said to have asymptotic density $d(A)$ if:

$$d(A)=\lim_{i,j,\ell\to\infty}\frac{1}{ij\ell}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}\chi\ A(K).$$

A triple sequence (real or complex) can be defined as a function $x:\mathbb{N}\times\mathbb{N}\times\mathbb{N}\to\mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural numbers, real numbers, and complex numbers respectively. The different types of notions of the triple sequence were introduced and investigated at the initial by (Şahiner *et al*., 2007; Sahiner and Tripathy, 2008; Esi, 2014; Esi and Catalbas, 2014; Esi and Savas, 2015; Esi *et al*., 2016; Datta *et al*., 2013; Subramanian and Esi, 2015; Esi *et al*., 2022, Debnath *et al*., 2015) and many others.

A triple sequence $x=(x_{mnk})$ is said to be triple analytic if:

$$\sup_{m,n,k}\left|x_{mnk}\right|^{\frac{1}{m+n+k}}<\infty.$$

The space of all triple analytic sequences is usually denoted by Λ^3.

Definitions and Preliminaries

Throughout the paper, $\mathbb{R}^3$ denotes the real three-dimensional case with the metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3$; $m, n, k \in \mathbb{N}^3$. The following definition is obtained.

Definition 1

Let f be a continuous function defined on the closed interval [0,1]. A triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operators $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ of real numbers and A = $(a_{i,j,\ell,m,n,k})$ be a non-negative RH-regular summability matrix is said to be rough statistically A-summable to $f(x)$ if for every $\epsilon > 0$:

$$\delta_2\left\{(i,j,\ell)\in\mathbb{N}^3:\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Ax)-f(x)\right|\geq r+\epsilon\right\}\in I=0,$$

i.e.:

$$P-\lim_{mnk}\frac{1}{mnk}$$
$$\left|\left\{i\leq m,j\leq n,\ell\leq k:\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Ax)-f(x)\right|\geq r+\epsilon\right\}\right|=0,$$

where, $(Ax)_{ij\ell}$ is as in (1.1).

A Korovkin-Type Approximation Theorem

Let $C_B(K)$ the space of all continuous and bounded real-valued functions on $K = [0, \infty) \times [0, \infty) \times [0, \infty)$. This space is equipped with the supremum norm:

$$\|f\|=\sup_{(x,y,z)\in K}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left|f,(x,y,z)\right|,\left(f\in C_B(K)\right).$$

Consider the triple space of $H_w(K)$ of all real-valued functions of Chlodowsky type (*λ, q*)-Bernstein Stancu operators of *f* on *K* satisfying:

$$\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;u,v,w)-B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x,y,z)\right|$$
$$\leq w\left(\left|\frac{u}{1+u}-\frac{x}{1+x}\right|,\left|\frac{u}{1+u}-\frac{y}{1+y}\right|,\left|\frac{w}{1+w}-\frac{z}{1+z}\right|\right)$$

where, *w* be a function of the type of the modulus of continuity given by, for $\delta, \delta_1, \delta_2, \delta_3 > 0$:

(1) *w* is non-negative increasing function on *K* with respect to $\delta_1, \delta_2, \delta_3$
(2) $w(\delta, \delta_1+\delta_2+\delta_3) \leq w(\delta, \delta_1) + w(\delta, \delta_2) + w(\delta, \delta_3)$
(3) $w(\delta_1+\delta_2+\delta_3, \delta) \leq w(\delta_1, \delta) + w(\delta_2, \delta) + w(\delta_3, \delta)$
(4) $\lim_{\delta_1,\delta_2\,\delta_3\to 0} w(\delta_1, \delta_2, \delta_3) = 0$

The Chlodowsky type (*λ, q*)-Bernstein Stancu operators $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Ax)\in H_w(K)$ satisfies the inequality:

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left|\left(f,(x,y,z)\right)\right|\leq B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,(0,0,0)\right)+w(1,1,1),x,y,z\geq 0$$

and hence it is bounded on *K*. Therefore:

$$H_w(K)\subset C_B(K).$$

We also use the following Chlodowsky type (*λ, q*)-Bernstein Stancu operators of test functions:

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{000},(u,v,w)\right)=1,B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{111},(u,v,w)\right)=\frac{u}{1+u},$$
$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{222},(u,v,w)\right)=\frac{u}{1+u},B_{mnk}\left(f_{333},(u,v,w)\right)=\frac{w}{1+w}$$

and:

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{444},(u,v,w)\right)=\left(\frac{u}{1+u}\right)^2+\left(\frac{v}{1+u}\right)^2+\left(\frac{w}{1+w}\right)^2.$$

Results

Theorem 1

Let f be a continuous function defined on the closed interval [0,1]. A triple sequence of Chlodowsky type (*λ, q*)-Bernstein Stancu operators $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ of real numbers from $H_w(K)$ into C_B (K) and let $A = (a_{i,j,\ell,m,n,k})$ be a nonnegative RH-regular summability matrix. Assume that the following conditions hold:

$$B^{st_2}-\lim\left\|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{rst}\right)-f_{rst}\right\|=0,r,s,t=0,1,2,3...$$

Then, for any f ∈ $H_w(K)$:

$$B^{st_2}-\lim\left\|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f)-f_{rst}\right\|=0. \quad (4.2)$$

Proof

Assume that (4.1) holds. Let $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,(x,y,z)\right)\in H_w(K)$ and $f(x, y, z) \in K$ be fixed.

Since $B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,(u,v,w)\right)\in H_w(K)$ for all $f(u,v,w)\in K$ be fixed, we write:

$$\left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,(u,v,w)\right)-B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,(x,y,z)\right)\right|$$
$$\leq r+\epsilon+\frac{2N}{\delta^2}\left\{\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2+\left(\frac{u}{1+u}-\frac{y}{1+y}\right)^2+\left(\frac{w}{1+w}-\frac{z}{1+z}\right)^2\right\}$$

where, $N:=\|f\|$. Using the linearity of Chlodowsky type (λ, q)-Bernstein Stancu operators $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f;x,y,z\right)\right)$, we obtain:

$$\left|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,(x,y,z)\right)-f(x,y,z)\right|$$
$$\leq r+\epsilon+C\left|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{000},(x,y,z)\right)-f(x,y,z)\right|+$$
$$C\left|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{111},(x,y,z)\right)-f(x,y,z)\right|+$$
$$C\left|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{222},(x,y,z)\right)-f(x,y,z)\right|+$$
$$C\left|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{333},(x,y,z)\right)-f(x,y,z)\right|$$

where:

$$c:=\max\left\{r+\epsilon+N+\frac{2N}{\delta^2}\left(\left(\frac{A}{1+A}\right)^2+\left(\frac{B}{1+B}\right)^2+\left(\frac{C}{1+C}\right)^2\right),\right.$$
$$\left.\frac{6N}{\delta^2}\left(\frac{A}{1+A}\right)\frac{6N}{\delta^2}\left(\frac{B}{1+B}\right),\frac{6N}{\delta^2}\left(\frac{C}{1+C}\right),\frac{2N}{\delta^2}\right\}.$$

Then, taking supremum over $f(x,y,z)\in K$ we get:

$$\left\|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f)-f\right\|$$
$$\leq r+\epsilon+C\sum_{r,s,t=0}^{3}\left\|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{rst}\right)-f_{rst}\right\| \quad (4.3)$$

For a given $\rho>0$, choose $r+\epsilon>0$ such that $(r+\epsilon)<\rho$. Then, for each $r,s,t=0,1,2,3$, setting:

$$U:=\left\{(i,j,\ell):\left\|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f)-f\right\|\geq\rho\right\},$$
$$U'_{rst}:=\left\{(i,j,\ell):\left\|\sum_{(m,n,k)}a_{ij\ell,m,n,k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{rst}\right)-f_{rst}\right\|\geq\frac{\rho-(r+\epsilon)}{6C}\right\},$$
$$(r,s,t=0,1,2,3),$$

it follows that (4.3) that:

$$U\subset\bigcup_{r,s,t=0}^{3}U_{rst}$$

which gives, for all $(i,j,\ell)\in N^3$:

$$\delta_2(U)\leq\sum_{r,s,t=0}^{3}\delta_2\left(U_{rst}\right)$$

From (4.1), we obtain (4.2). This completes the proof.

If we take $A=I$, which is the identity matrix we get the following statistical version of Theorem 1.

Corollary 1

Let f be a continuous function defined on the closed interval [0,1]. A triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operators $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ of real numbers from $H_w(K)$ into C_B (K). Assume that the following conditions hold:

$$B^{st_2}-\lim\left\|B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{rst}\right)-f_{rst}\right\|=0, r,s,t=0,1,2,3... \quad (4.4)$$

Then, for any $f\in H_w(K)$:

$$B^{st_2}-\lim\left\|\sum_{(m,n,k)}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f)-f\right\|=0 \quad (4.5)$$

In Corollary 1, if the statistical convergence ([C,1,1] statistical convergence) replace with Pringsheim convergence, we obtain the following classical version of Theorem 1.

Corollary 2

Let f be a continuous function defined on the closed interval [0,1]. A triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu operators $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ of real numbers from $H_w(K)$ into $C_B(K)$. Assume that the following conditions hold:

$$P-\lim\left\|\sum_{(m,n,k)}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{rst}\right)-f_{rst}\right\|=0, r,s,t=0,1,2,3,... \quad (4.6)$$

Then, for any $f\in H_w(K)$:

$$P-\lim\left\|\sum_{(m,n,k)}B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f)-f\right\|=0 \quad (4.7)$$

Remark 1. We now show that our result Theorem 1 is stronger than its classical version Corollary 2 and statistical version Corollary 1. To see this first consider the following Bleimann, Butzer, and Hahn operators of three variables of Chlodowsky type (λ, q)-Bernstein Stancu operators $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ are:

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,(x,y,z)\right)=\frac{1}{(1+x)^m(1+y)^n(1+x)^k}\sum_{i=0}^{m}\sum_{i=0}^{n}\sum_{\ell=0}^{k}B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,\left(\frac{i}{m-i+1},\frac{i}{n-j+1},\frac{\ell}{k-\ell+1}\right)\right)\binom{m}{i}\binom{n}{j}\binom{k}{\ell}x^iy^jz^\ell, \tag{4.8}$$

where, $f\in H_w(K)$ and $K = [0,\infty)\times[0,\infty)\times[0,\infty)$. We have:

$$\begin{aligned}
&B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{000},(x,y,z)\right)=1, B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{111},(x,y,z)\right)=\frac{m}{m+1}\frac{x}{1+x},\\
&B_{mnk},\left(f_{222},(x,y,z)\right)=\frac{n}{n+1}\frac{y}{1+y}, B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{333},(x,y,z)\right)=\frac{k}{k+1}\frac{z}{1+z},\\
&B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f_{444},(x,y,z)\right)=\frac{m(m-2)(m-1)}{m+1}\left(\frac{x}{1+x}\right)^3+\frac{m(m-1)}{(m+1)^2}\\
&\left(\frac{x}{1+x}\right)^2+\frac{m}{(m+1)^2}\frac{x}{1+x}+\frac{n(n-2)(n-1)}{(n+1)^2}\left(\frac{y}{1+y}\right)^3\\
&+\frac{n(n-1)}{(n+1)^2}\left(\frac{y}{1+y}\right)^2+\frac{n}{(n+1)^2}\frac{y}{1+y}\\
&+\frac{k(k-2)(k-1)}{(k+1)^2}\left(\frac{z}{1+z}\right)^3+\frac{k(k-1)}{(k+1)^2}\left(\frac{z}{1+z}\right)^2+\frac{k}{(k+1)^2}\frac{z}{1+z}.
\end{aligned} \tag{4.9}$$

Now take A = [C,1,1,1] and define a triple sequence *u*: = (u_{mnk}) by:

$$u_{mnk}=(-1)^{m+n+k} \tag{4.10}$$

we observe that:

$$B^{st_2}-\lim C\,[1,1,1](u)=0 \tag{4.11}$$

However, the Chlodowsky type (λ, q)-Bernstein Stancu operators $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ of the triple sequence of u is not P-convergent and statistical convergent. Now using (4.10) and (4.11), we define the following double-positive linear operators on $H_w(K)$ as follows:

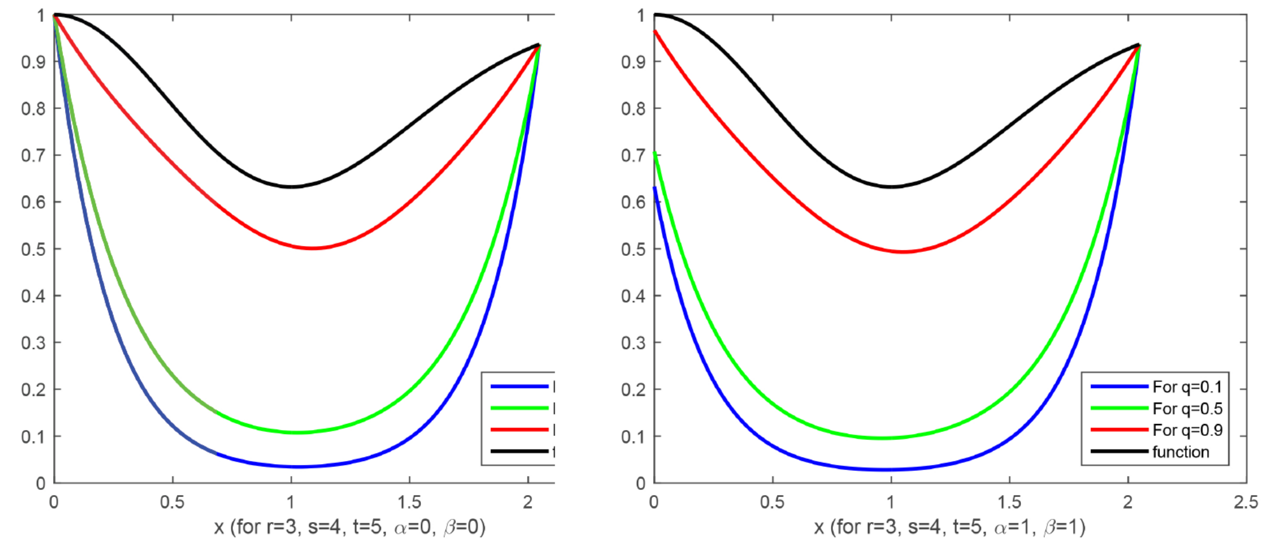
$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,(x,y,z)\right)=\left(1+u_{mnk}\right)B^{\alpha,\beta}_{(r,s,t),\lambda,q}\left(f,(x,y,z)\right) \tag{4.12}$$

Then, observe that the Chlodowsky type (λ, q)-Bernstein Stancu operators $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x)\right)$ of a triple sequence $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}\right)$ defined by (4.12) satisfy all hypotheses of Theorem 1. Hence, by (4.9) and (4.11), we have, for all f ∈ $H_w(K)$:

$$B^{st_2}-\lim\left\|\sum_{(m,n,k)}a_{ij\ell,\,m,n,k}\;B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f)-f\right\|=0$$

Since u is not P-convergent and statistical convergent, the sequence $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f)\right)$ cannot uniformly converge to f on K or statistical sense.

Example 1 with the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators (1.2) to the function $f(x) = 1 - x^2e^{-x2}$ under different parameters.



(a) (b)

Fig. 1: Chlodowsky type (λ, q)-Bernstein-Stancu operators

From Fig. 1(a), it can be observed that as the value the q approaches towards 1 provided $0 < q \leq 1$, Chlodowsky type (λ, q)-Bernstein-Stancu operators given by (1.2) converges towards the function $f(x) = 1 - x^2e^{-x2}$. From Fig. 1(a), it can be observed that for $\alpha = \beta = 0$, as the value the (r, s, t) increases, Chlodowsky type (λ, q)-Bernstein-Stancu operators given by (1.2) converges towards the function. Similarly from Fig. 1(b), it can be observed that for $\alpha = \beta = 1$, as the value the q approaches towards 1 or something else provided $0<q\leq1$, Chlodowsky type (λ, q)Bernstein-Stancu operators given by (1.2) converges towards the function. From Fig. 1(b), it can be observed that as the value the [r, s, t] increases, Chlodowsky type (λ, q)-Bernstein-Stancu operators given by $f(x) = 1-x^2e^{-x2}$ converges towards the function.

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Author's Contributions

All authors equally contributed in this study.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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